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Varieties of Rings and Groups

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1. INTRODUCTION

The author pointed out group axioms in [1] for the collection of unipotent 3×3 matrix groups $U_3(R) = \{I + A \mid A$ strictly upper triangular over $R\}$ over arbitrary nonassociative rings R, citing as a natural consequence the equivalence between the theories of these groups and nonassociative rings. We present here a different and more facile characterization of these groups.

If this equivalence were such that R has a finite basis for its laws provided $U_3(R)$ has, then the fact that $U_3(R)$ is nilpotent implies that R has a finite basis [2]. However, it is not clear whether this is true, since the equivalence referred to does not translate laws for R into laws for $U_3(R)$.

Motivated by this new characterization however, we obtain the following equivalence.

MAIN THEOREM. *Let R_∞ and F_∞ denote any free nonassociative ring and group of countable rank respectively and if*

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and

$$C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

then every n-variable word $r(x_1, \ldots, x_n)$ of R_∞ corresponds to a $2n + 2$ variable word $w(y_1, \ldots, y_{2n}, y_{2n+1}, y_{2n+2})$ of F_∞ such that $r(x_1, \ldots, x_n)$ is a law for a *nonassociative ring R if and only if* $w(g_1, g_2, \ldots, g_{2n}, A, B) = 1$ *for all* $g_1, \ldots, g_{2n}$ *in* $\mathrm{gp}\{C, U_3(R)\}$.

It is still uncertain whether for R associative $w(y_1, \ldots, y_{2n}, y_{2n+1}, y_{2n+2})$ is a law for $G(R) = \mathrm{gp}\{A, B, C, U_3(R)\}$ or whether $G(R)$ has a finite basis for its law if and only if R has.

2. Group Characterization of 3×3 Unipotent Groups Over Rings

We prove:

Theorem 1. *G is isomorphic to a 3×3 unipotent group $U_3(R)$ over a nonassociative ring R*[1] *if and only if G has two normal subgroups N_1, N_2 such that:*

(a) $G = N_1 \cdot N_2$

(b) *N_i is abelian* $(i = 1, 2)$

(c) *N_i is the direct product of $N_1 \cap N_2$ and a subgroup $A_i \cong N_1 \cap N_2$,* $(i = 1, 2)$

Proof. That a unipotent group $U_3(R)$ satisfies (a), (b), and (c) is left to the reader to verify.

Assume that G satisfies (a), (b), and (c). Then let $I(A_i)$ denote the set of isomorphisms of $N_1 \cap N_2 = A_0$ onto A_i $(i = 1, 2)$. From $\phi_i \in I(A_i)$, $(i = 1, 2)$, and G, a ring $R(\phi_1, \phi_2)$ is constructed so that $U_3(R(\phi_1, \phi_2)) \cong G$. To accomplish this let $R^+(\phi_1, \phi_2) = A_0$ and define multiplication $\boxtimes$ by

$$x \boxtimes y = [x^{\phi_1}, y^{\phi_2}],\ x, y \in A_0, \qquad \text{where} \qquad [u, v] = u^{-1}v^{-1}uv.$$

We shall denote ring addition by $\boxplus$.

$R(\phi_1, \phi_2)$ is closed under multiplication since $G' \subseteq A_0$. The distributive laws follow easily.

We now show that an isomorphism between $U_3(R(\phi_1, \phi_2))$ and G is given by the map δ, which is defined by

$$\begin{pmatrix} 1 & x_2 & x_3 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\delta} x_1^{\phi_1} x_2^{\phi_2} x_3 .$$

First note that δ is onto, and one to one since every element of G can be

[1] By "non-associative" we mean "not necessarily associative".

written uniquely as xyz where $x \in A_1$, $y \in A_2$, $z \in A_0$. Since x_3 is central and $G' \leqslant A_0$

$$x_1^{\phi_1} x_2^{\phi_2} x_3 \cdot y_1^{\phi_1} y_2^{\phi_2} y_3 = x_1^{\phi_1} y_1^{\phi_1} x_2^{\phi_2} [x_2^{\phi_2}, y_1^{\phi_1}]\, y_2^{\phi_2} x_3 y_3$$

$$= (x_1 y_1)^{\phi_1} (x_2 y_2)^{\phi_2} [x_2^{\phi_2}, y_1^{\phi_1}]\, x_3 y_3$$

$$= (x_1 \boxplus y_1)^{\phi_1}, (x_2 \boxplus y_2)^{\phi_2} (x_2 \boxtimes y_1 \boxplus x_3 \boxplus y_3)$$

$$= \left[\begin{pmatrix} 1 & x_2 & x_3 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & y_2 & y_3 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{pmatrix}\right]^{\delta}$$

Therefore δ is a homomorphism and thus the proof is completed.

Some obvious consequences follow:

COROLLARY 1. *If G satisfies* (a), (b), *and* (c) *then*

(i) $N_1 \cap N_2 = R^+$ *and* $G \cong U_3(R)$, *for some nonassociative ring* R

(ii) G *is a central extension of* $N_1 \cap N_2$ *by* $A_1 \times A_2$

(iii) G *is a splitting extension of* N_i *by* A_j $(i \neq j,\ i = 1, 2)$.

3. Proof of Main Theorem

In proving the main theorem we may assume for convenience that both R_∞ and F_∞ are freely generated by the same set S.

Given a ring-word $r(x_1, \dots, x_n)$, we define the $n + 2$ variable word $v(x_1, \dots, x_n, y_{2n+1}, y_{2n+2})$ in F_∞ by replacing ring addition with group multiplication and replacing ring multiplication $x_i \cdot x_j$ by $[x_i^{y_{2n+1}}, x_j^{y_{2n+2}}]$ for fixed generators $y_{2n+1}, y_{2n+2} \in S$ where y_{2n+1}, y_{2n+2} are distinct from each other as well as from $x_1, \dots, x_n$ ($g^t = t^{-1}gt$ for group elements g and t.)

By Theorem 1, $U_3(R)$ contains normal subgroups N_1, N_2 satisfying (a), (b) and (c). In fact

$$N_1 = \left\{ \begin{pmatrix} 1 & y & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y \in R \right\}$$

$$N_2 = \left\{ \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, \in R \right\}$$

$$A_1 = \left\{ \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad | \, x \in R \right\}$$

$$A_2 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \quad | \, x \in R \right\}$$

If we define the isomorphisms ϕ_1, ϕ_2 in $I(A_1)$, $I(A_2)$ respectively by

$$\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{\phi_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{\phi_2} = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

it follows easily that $R \cong R(\phi_1, \phi_2)$.

Therefore since

$$\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{A} = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{\phi_1}$$

$$\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{B} = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{\phi_2}$$

$r(x_1, \ldots, x_n)$ is a law for R if and only if $v(g_1, \ldots, g_n, A, B) = 1$ for all $g_1, \ldots, g_n$ in $N_1 \cap N_2$.

However, not only does $N_1 \cap N_2$ contain the commutator subgroup of $\mathrm{gp}\{C, U_3(R)\}$ but every element is a commutator of C with an element of $U_3(R)$.

$$\left(\text{i.e.} \quad \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \left[C, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \right] \right)$$

Define $w(y_1, \ldots, y_{2n}, y_{2n+1}, y_{2n+2})$ from $v(x_1, \ldots, x_n, y_{2n+1}, y_{2n+2})$ by replacing each x_i ($i \leqslant n$) with $[y_{2i-1}, y_{2i}]$ where $y_{2i-1}, y_{2i} \in S$ and $y_1, \ldots, y_{2n}$, y_{2n+1}, y_{2n+2} are all distinct. Then the condition $v(g_1, \ldots, g_n, A, B) = 1$ for all values of $g_1, \ldots, g_n$ in $N_1 \cap N_2$ is equivalent to $w(g_1, \ldots, g_{2n}, A, B) = 1$ for all values of $g_1, \ldots, g_{2n}$ in $\mathrm{gp}\{C, U_3(R)\}$.

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